Problem Set 1

This problem set is longer than most of the problem sets we will discuss this term. Since this is the first week of the semester, don't worry if you don't have time to solve the whole PS before the sessions or if parts of the PS were not yet covered in the lecture. We will go over things slowly and finish the discussion only next week (Nov 11 and Nov 13).

The Solow model

Consider the standard Solow model, described by the following system of equations:

$$\begin{aligned} \dot{K}_t &= sY_t - \delta K_t, \\ \dot{L}_t &= nL_t, \\ Y_t &= F(K_t, A_tL_t), \\ \dot{A}_t &= aA_t. \end{aligned}$$

a) State carefully in words the economic meaning of each of these equations.

Define the capital stock per efficiency unit of labor $k_t = \frac{K_t}{A_tL_t}$. We assume that the production function F has constant returns to scale. Therefore the intensive form production function is given by:

$$f(k_t) = F(\frac{K_t}{A_tL_t}, 1) \quad \text{with} \quad F(K_t, A_tL_t) = F(\frac{K_t}{A_tL_t}, 1)A_tL_t = f(k_t)A_tL_t.$$

b) Derive the dynamic equation of the capital stock k_t per efficiency unit of labor.

c) Assume that the economy is in steady state ($\dot{k}_t = 0$). Compute growth rates of K_t, Y_t and $C_t = (1-s)Y_t$.

   Hints: The growth rate of a variable x_t is defined as $\gamma_x(t) = \frac{\dot{x}_t}{x_t}$. Taking logarithms and then differentiating with respect to time is generally a good strategy to compute growth rates. It also helps to use the intensive form production function.

d) Use the Solow diagram to depict this model's steady state as the intersection of the savings function $sf(k_t)$ and the capital widening function given by $(a + \delta + n)k_t$.

e) Assuming the Cobb-Douglas production function $F(K_t, A_tL_t) = K_t^{\alpha}(A_tL_t)^{1-\alpha}$, find the intensive form production function $f(k_t)$. Then use it to compute the steady state(s) depicted in d).

A more exotic production function

In most cases we use a Cobb-Douglas production function in the Solow model. The Cobb-Douglas production function satisfies the conditions that guarantee a unique stable steady state and the existence of a balanced growth path for all values of α. To get a better understanding why these conditions are important, we now consider a Solow model with a different production function which might not satisfy all conditions. From now on, assume that the production function F is given by:

$$F(K_t, A_t L_t) = \ln\left(1 + \frac{K_t}{A_t L_t}\right) A_t L_t$$

f) Does this production function have constant returns to scale in K and L?

g) Determine the intensive form production function $f(k_t)$.

h) Is the intensive form production function increasing in k_t? Does have it have decreasing marginal product in k_t? Does the intensive form production function satisfy the Inada conditions? Give some intuition for your answer and why this is might be important in the Solow model.

i) Show that the dynamic equation of the capital stock per efficiency unit derived in b) is given by the following equation:

$$\dot{k}_t = s \ln(1 + k_t) - (n + a + \delta) k_t \tag{1}$$

Graphical approach

The differential equation (1) describes the dynamic evolution of the economy in our Solow model. This is a special type of differential equation. As you can see, time t only appears as a subscript of k_t here but not directly in the equation.[1] This property allows us to analyze it using a graphical approach. How does the graphical approach work? First, the fact that t does not appear in the differential equation allows us to drop the time index and set $\dot{k} = 0$ to solve for the steady states of the equation. Steady states are values of k_t, for which k_t does not change. This means, if k_t takes a steady state value once, it will remain there for ever.

Next, we determine for which values of k_t does equation (1) imply that it is increasing ($\dot{k}_t$ positive) or decreasing ($\dot{k}_t$ negative). Note that, whenever the right hand side of equation (1) is positive, $\dot{k}_t$ is positive such that k_t must be increasing in t. On the other hand, if the right hand side of equation (1) is negative k_t must be declining in t. Equation (1) is called *locally stable* around steady state k^*, if, starting in a small neighborhood around the steady state (k_0 close enough to k^*), k_t converges to the steady state as $t \to \infty$. Define $g(k_t) = s \ln(1 + k_t) - (n + a + \delta) k_t$. Then a sufficient condition for local stability is $g'(k^*) < 0$.

Follow the steps outlined above to answer the following questions:

j) Depict equation (1) in a diagram by drawing $\dot{k}_t$ on the vertical, k_t on the horizontal axis. Find the steady state(s) in the diagram. How can you tell from the graph whether this differential equation's steady state(s) is (are) locally stable? Explain the intuition behind the analytical condition $g'(k^*) < 0$.

[1] Compare this to differential equations in PS 0 in which time shows up as an explicit variable. Differential equations that do not directly depend on the argument t are called *autonomous*. They occur frequently in economics when the role of time is solely to mark the accumulation or decay of some other variable, for example the capital stock or the money supply.

k) For which parameter values does a positive stable steady state $k^* > 0$ exist? How does this answer relate to question h)?

Model dynamics

From now on assume that the condition derived in question k) is satisfied. We can use the diagram we have just drawn to analyze how the model economy evolves over time starting at a given level of capital.

l) Consider an economy where k_t is in the steady state and suppose the rate of population growth falls permanently, from say, n to n'. Determine how the steady state values of output y_t and capital k_t change. Show your answer using both the Solow diagram as well as a graph of k_t and y_t as a function of time.

m) Consider again an economy where k_t is in the steady state. Suppose that a meteor strike destroys half of the capital in the economy. Assume that the population is warned in advance and takes shelter, so everyone survives and L_t is not affected. How does this affect steady state values of k_t and y_t? Show your answer using both the Solow diagram as well as a graph of k_t and y_t as a function of time.